

# Direct Method on Stochastic Maximum Principle for Optimization with Recursive Utilities

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**Abstract.** We obtain the variational equations for backward stochastic differential equations in recursive stochastic optimal control problems, and then get the maximum principle which is novel. The control domain need not be convex, and the generator of the backward stochastic differential equation can contain  $z$ .

**Key words.** Backward stochastic differential equations, Recursive stochastic optimal control, Maximum principle, Variational equation

**AMS subject classifications.** 93E20, 60H10, 49K45

## 1 Introduction

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and let  $W$  be a  $d$ -dimensional Brownian motion. The filtration  $\{\mathcal{F}_t : t \geq 0\}$  is generated by  $W$ , i.e.,

$$\mathcal{F}_t := \sigma\{W(s) : s \leq t\} \vee \mathcal{N},$$

where  $\mathcal{N}$  is all  $P$ -null sets. Let  $U$  be a set in  $\mathbb{R}^k$  and  $T > 0$  be a given terminal time. Set

$$\mathcal{U}[0, T] := \{(u(s))_{s \in [0, T]} : u \text{ is progressively measurable, } u(s) \in U \text{ and } E[\int_0^T |u(s)|^\beta ds] < \infty \text{ for all } \beta > 0\},$$

where  $U$  is called the control domain and  $\mathcal{U}[0, T]$  is called the set of all admissible controls. In fact, we just need  $E[\int_0^T |u(s)|^{\beta_0} ds] < \infty$  for some  $\beta_0 > 0$ . For simplicity, we do not explicitly give this  $\beta_0$  in this paper. Consider the following state equation:

$$\begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where  $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,  $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^{n \times d}$ . The cost functional is defined by

$$J(u(\cdot)) = E[\phi(x(T)) + \int_0^T f(t, x(t), u(t))dt], \quad (1.2)$$

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where  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$ . The classical stochastic optimal control problem is to minimize  $J(u(\cdot))$  over  $\mathcal{U}[0, T]$ . If there exists a  $\bar{u} \in \mathcal{U}[0, T]$  such that

$$J(\bar{u}(\cdot)) = \inf_{u \in \mathcal{U}[0, T]} J(u(\cdot)),$$

$\bar{u}$  is called an optimal control.  $\bar{x}(\cdot)$ , which is the solution of state equation (1.1) corresponding to  $\bar{u}$ , is called an optimal trajectory. The maximum principle is to find the necessary condition for the optimal control  $\bar{u}$ .

The method for deriving the maximum principle is the variational principle. When  $U$  is not convex, we use the spike variation method. More precisely, let  $\varepsilon > 0$  and  $E_\varepsilon \subset [0, T]$  with  $|E_\varepsilon| = \varepsilon$ , define

$$u^\varepsilon(t) = \bar{u}(t)I_{E_\varepsilon^c}(t) + uI_{E_\varepsilon}(t),$$

where  $u \in U$ . This  $u^\varepsilon$  is called a spike variation of the optimal control  $\bar{u}$ . For deriving the maximum principle, we only need to use  $E_\varepsilon = [s, s + \varepsilon]$  for  $s \in [0, T - \varepsilon]$  and  $\varepsilon > 0$ . The difficulty of the classical stochastic optimal control problem is the variational equation for  $x(\cdot)$ , which is completely different from the deterministic optimal control problem. Peng [12] first considered the second-order term in the Taylor expansion of the variation and obtained the maximum principle for the classical stochastic optimal control problem.

Consider the following backward stochastic differential equation (BSDE for short):

$$y(t) = \phi(x(T)) + \int_t^T f(s, x(s), y(s), z(s), u(s))ds - \int_t^T z(s)dW(s), \quad (1.3)$$

where  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ . Pardoux and Peng [11] first obtained that the BSDE (1.3) has a unique solution  $(y(\cdot), z(\cdot))$ . Duffie and Epstein [3] introduced the notion of recursive utilities in continuous time, which is a kind of BSDE with  $f$  independent of  $z$ . In [4, 5], the authors extended the recursive utility to the case where  $f$  contains  $z$ . The term  $z$  can be interpreted as an ambiguity aversion term in the market (see [1]).

When  $f$  is independent of  $(y, z)$ , it is easy to check that  $y(0) = E[\phi(x(T)) + \int_0^T f(t, x(t), u(t))dt]$ . So it is natural to extend the classical stochastic optimal control problem to the recursive case. Consider the control system which contains equations (1.1) and (1.3). Define the cost functional

$$J(u(\cdot)) = y(0). \quad (1.4)$$

The recursive stochastic optimal control problem is to minimize  $J(u(\cdot))$  in (1.4) over  $\mathcal{U}[0, T]$ . When the control domain  $U$  is convex, the local maximum principle for this problem can be found in [2, 7, 13, 17, 19, 21] and the references therein. In this paper, the control domain  $U$  is not necessarily convex, we must obtain the global maximum principle by the spike variation method.

One direct method for treating this problem is to consider the second-order terms in the Taylor expansion of the variation for the BSDE (1.3) as in [12]. When  $f$  depends nonlinearly on  $z$ , there are two major difficulties (see [24]): (i) What is the second-order variational equation for the BSDE (1.3), which is not the one similar in [12]. (ii) How to get the second-order adjoint equation which seems to be unexpectedly complicated due to the quadratic form with respect to the variation of  $z$ .

Based on these difficulties, Peng [15] proposed the following open problem in page 269:

“The corresponding ‘global maximum principle’ for the case where  $f$  depends nonlinearly on  $z$  is open, except for some special case.”

Recently, a new method for treating this problem is to see  $z(\cdot)$  as a control and the terminal condition  $y(T) = \phi(x(T))$  as a constraint, then use the Ekeland variational principle to obtain the maximum principle. This idea was used in [9, 10] for studying the backward linear-quadratic optimal control problem, and then was used in [20, 24] for studying the recursive stochastic optimal control problem. But the maximum principle contains unknown parameters.

In this paper, we overcome the two major difficulties in the above direct method. The second-order variational equation for the BSDE (1.3) and the maximum principle have been obtained. The main difference of the variational equations with the ones in [12] lies in the term  $\langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t)$  (see equation (3.1) in Section 3 for the definition of  $p(t)$ ) in the variation of  $z$ , which is  $O(\varepsilon)$  for any order expansion of  $f$ . So it is not helpful to use the second-order Taylor expansion for treating this term. Moreover, we also obtain the structure of the variation for  $(y, z)$  and the variation for  $x$ . Based on this, we can get the second-order adjoint equation. Due to the term  $\langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t)$  in the variation of  $z$ , our global maximum principle is novel and different from the one in [20, 24], which completely solves Peng’s open problem. Furthermore, our maximum principle is stronger than the one in [20, 24] (see Example 3.8).

This paper is organized as follows. In Section 2, we give some basic results and the idea for the variation of BSDE. The variational equations for BSDE and the maximum principle have been obtained in Section 3. In Section 4, we obtain the maximum principle for the control system with state constraint.

## 2 Preliminaries and idea for variation of BSDE

The results of this part can be found in [12, 25]. For the simplicity of presentation, we suppose  $d = 1$ . We need the following assumption:

**(A1)**  $b, \sigma$  are twice continuously differentiable with respect to  $x$ ;  $b, b_x, b_{xx}, \sigma, \sigma_x, \sigma_{xx}$  are continuous in  $(x, u)$ ;  $b_x, b_{xx}, \sigma_x, \sigma_{xx}$  are bounded;  $b, \sigma$  are bounded by  $C(1 + |x| + |u|)$ .

Let  $\bar{u}(\cdot)$  be the optimal control for the cost function defined in (1.2) and let  $\bar{x}(\cdot)$  be the corresponding solution of equation (1.1). Similarly, we define  $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ . Set

$$\begin{aligned} b(\cdot) &= (b^1(\cdot), \dots, b^n(\cdot))^T, \quad \sigma(\cdot) = (\sigma^1(\cdot), \dots, \sigma^n(\cdot))^T, \\ b(t) &= b(t, \bar{x}(t), \bar{u}(t)), \quad \delta b(t) = b(t, \bar{x}(t), u) - b(t), \end{aligned} \tag{2.1}$$

similar for  $b_x(t), b_{xx}^i(t), \delta b_x(t), \delta b_{xx}^i(t), \sigma(t), \sigma_x(t), \sigma_{xx}^i(t), \delta\sigma(t), \delta\sigma_x(t)$  and  $\delta\sigma_{xx}^i(t)$ ,  $i \leq n$ , where  $b_x = (b_{x_j}^i)_{i,j}$ . Let  $x_i(\cdot)$ ,  $i = 1, 2$ , be the solution of the following stochastic differential equations (SDEs for short):

$$\begin{cases} dx_1(t) = b_x(t)x_1(t)dt + \{\sigma_x(t)x_1(t) + \delta\sigma(t)I_{E_\varepsilon}(t)\}dW(t), \\ x_1(0) = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} dx_2(t) = \{b_x(t)x_2(t) + \delta b(t)I_{E_\varepsilon}(t) + \frac{1}{2}b_{xx}(t)x_1(t)x_1(t)\}dt \\ \quad + \{\sigma_x(t)x_2(t) + \delta\sigma_x(t)x_1(t)I_{E_\varepsilon}(t) + \frac{1}{2}\sigma_{xx}(t)x_1(t)x_1(t)\}dW(t), \\ x_2(0) = 0, \end{cases} \quad (2.3)$$

where  $b_{xx}(t)x_1(t)x_1(t) = (\text{tr}[b_{xx}^1(t)x_1(t)x_1(t)^T], \dots, \text{tr}[b_{xx}^n(t)x_1(t)x_1(t)^T])^T$  and similar for  $\sigma_{xx}(t)x_1(t)x_1(t)$ .

**Theorem 2.1** *Suppose (A1) holds. Then, for any  $\beta \geq 1$ ,*

$$E[\sup_{t \in [0, T]} |x^\varepsilon(t) - \bar{x}(t)|^{2\beta}] = O(\varepsilon^\beta), \quad (2.4)$$

$$E[\sup_{t \in [0, T]} |x_1(t)|^{2\beta}] = O(\varepsilon^\beta), \quad (2.5)$$

$$E[\sup_{t \in [0, T]} |x_2(t)|^{2\beta}] = O(\varepsilon^{2\beta}), \quad (2.6)$$

$$E[\sup_{t \in [0, T]} |x^\varepsilon(t) - \bar{x}(t) - x_1(t)|^{2\beta}] = O(\varepsilon^{2\beta}), \quad (2.7)$$

$$E[\sup_{t \in [0, T]} |x^\varepsilon(t) - \bar{x}(t) - x_1(t) - x_2(t)|^{2\beta}] = o(\varepsilon^{2\beta}). \quad (2.8)$$

Moreover, we have the following expansion:

$$\begin{aligned} E[\phi(x^\varepsilon(T))] - E[\phi(\bar{x}(T))] &= E[\langle \phi_x(\bar{x}(T)), x_1(T) + x_2(T) \rangle] \\ &\quad + E[\frac{1}{2}\langle \phi_{xx}(\bar{x}(T))x_1(T), x_1(T) \rangle] + o(\varepsilon). \end{aligned} \quad (2.9)$$

From this result, we can simply write  $x_1(\cdot) = O(\sqrt{\varepsilon})$ ,  $x_2(\cdot) = O(\varepsilon)$  and

$$x^\varepsilon(\cdot) = \bar{x}(\cdot) + x_1(\cdot) + x_2(\cdot) + o(\varepsilon). \quad (2.10)$$

This equation is the variational equation for SDE (1.1) obtained by Peng in [12]. Furthermore, he proved that

$$E[\langle \phi_x(\bar{x}(T)), x_1(T) \rangle] = O(\varepsilon).$$

This is surprise, because  $x_1(T) = O(\sqrt{\varepsilon})$ . The reason is that the order  $\sqrt{\varepsilon}$  term is in the stochastic integral with respect to  $W$ . Notice that  $z$  in BSDE (1.3) is still in the stochastic integral with respect to  $W$ . If we combine the order  $\sqrt{\varepsilon}$  term and  $z$ , the other terms are  $O(\varepsilon)$ . Maybe we can get the variational equation. In Section 3, we will give a rigorous proof of this idea.

### 3 Variational equation for BSDE and maximum principle

#### 3.1 Peng's open problem

Suppose  $d = 1$  for the simplicity of presentation. The results for  $d > 1$  will be given in the next subsection.

We consider the control system: SDE (1.1) and BSDE (1.3). The cost function  $J(u(\cdot))$  is defined in (1.4). The control problem is to minimize  $J(u(\cdot))$  over  $\mathcal{U}[0, T]$ .

We need the following assumption:

(A2)  $f, \phi$  are twice continuously differentiable with respect to  $(x, y, z)$ ;  $f, Df, D^2f$  are continuous in  $(x, y, z, u)$ ;  $Df, D^2f, \phi_{xx}$  are bounded;  $\phi_x$  are bounded by  $C(1 + |x|)$ .

$Df$  is the gradient of  $f$  with respect to  $(x, y, z)$ ,  $D^2f$  is the Hessian matrix of  $f$  with respect to  $(x, y, z)$ .

Let  $\bar{u}(\cdot)$  be the optimal control and let  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$  be the corresponding solution of equations (1.1) and (1.3). Similarly, we define  $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot), u^\varepsilon(\cdot))$ .

In order to obtain the variational equation for BSDE (1.3), we introduce the following adjoint equations:

$$\begin{cases} -dp(t) = \{f_y(t)p(t) + [f_z(t)\sigma_x^T(t) + b_x^T(t)]p(t) + f_z(t)q(t) + \sigma_x^T(t)q(t) + f_x(t)\}dt - q(t)dW(t), \\ p(T) = \phi_x(\bar{x}(T)), \end{cases} \quad (3.1)$$

$$\begin{cases} -dP(t) = \{f_y(t)P(t) + [f_z(t)\sigma_x^T(t) + b_x^T(t)]P(t) + P(t)[f_z(t)\sigma_x(t) + b_x(t)] + \sigma_x^T(t)P(t)\sigma_x(t) \\ \quad + f_z(t)Q(t) + \sigma_x^T(t)Q(t) + Q(t)\sigma_x(t) + b_{xx}^T(t)p(t) + \sigma_{xx}^T(t)[f_z(t)p(t) + q(t)] \\ \quad + [I_{n \times n}, p(t), \sigma_x^T(t)p(t) + q(t)]D^2f(t)[I_{n \times n}, p(t), \sigma_x^T(t)p(t) + q(t)]^T\}dt - Q(t)dW(t), \\ P(T) = \phi_{xx}(\bar{x}(T)), \end{cases} \quad (3.2)$$

where

$$f_x(t) = f_x(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), \quad p(t) = (p^1(t), \dots, p^n(t))^T, \quad b_{xx}^T(t)p(t) = \sum_{i=1}^n p^i(t)b_{xx}^i(t), \quad (3.3)$$

similar for  $f_y(t), f_z(t), D^2f(t)$  and  $\sigma_{xx}^T(t)[f_z(t)p(t) + q(t)]$ . By the assumptions (A1) and (A2), it is easy to check that the above BSDEs have unique solutions  $(p(\cdot), q(\cdot))$  and  $(P(\cdot), Q(\cdot))$  respectively.

Applying Itô's formula to  $\langle p(t), x_1(t) + x_2(t) \rangle + \frac{1}{2}\langle P(t)x_1(t), x_1(t) \rangle$ , we can get the following lemma by simple calculation.

**Lemma 3.1** *We have*

$$\begin{aligned} & \langle p(T), x_1(T) + x_2(T) \rangle + \frac{1}{2}\langle P(T)x_1(T), x_1(T) \rangle = \langle p(t), x_1(t) + x_2(t) \rangle + \frac{1}{2}\langle P(t)x_1(t), x_1(t) \rangle \\ & + \int_t^T \{ \langle p(s), \delta b(s) \rangle + \langle q(s), \delta \sigma(s) \rangle + \frac{1}{2}\langle P(s)\delta \sigma(s), \delta \sigma(s) \rangle \} I_{E_\varepsilon}(s) \\ & - \langle f_x(s) + f_y(s)p(s) + f_z(s)[\sigma_x^T(s)p(s) + q(s)], x_1(s) + x_2(s) \rangle \\ & - \frac{1}{2}\langle \{f_y(s)P(s) + f_z(s)[\sigma_{xx}^T(s)p(s) + P(s)\sigma_x(s) + \sigma_x^T(s)P(s) + Q(s)] \\ & + [I_{n \times n}, p(s), \sigma_x^T(s)p(s) + q(s)]D^2f(s)[I_{n \times n}, p(s), \sigma_x^T(s)p(s) + q(s)]^T\}x_1(s), x_1(s) \rangle \} ds \\ & + \int_t^T \{ \langle p(s), \delta \sigma(s) \rangle I_{E_\varepsilon}(s) + \langle \sigma_x^T(s)p(s) + q(s), x_1(s) + x_2(s) \rangle \\ & + \langle \delta \sigma_x^T(s)p(s) + \frac{1}{2}P(s)\delta \sigma(s) + \frac{1}{2}P^T(s)\delta \sigma(s), x_1(s) \rangle \} I_{E_\varepsilon}(s) \\ & + \frac{1}{2}\langle [\sigma_{xx}^T(s)p(s) + P(s)\sigma_x(s) + \sigma_x^T(s)P(s) + Q(s)]x_1(s), x_1(s) \rangle \} dW(s) \\ & + \int_t^T \langle \delta \sigma_x^T(s)q(s) + \frac{1}{2}[\sigma_x^T(s)P(s) + \sigma_x^T(s)P^T(s) + Q(s) + Q^T(s)]\delta \sigma(s), x_1(s) \rangle I_{E_\varepsilon}(s) ds. \end{aligned} \quad (3.4)$$

By Theorem 2.1, it is easy to check that the last line of equation (3.4) is  $o(\varepsilon)$  and

$$\phi(x^\varepsilon(T)) = \phi(\bar{x}(T)) + \langle p(T), x_1(T) + x_2(T) \rangle + \frac{1}{2} \langle P(T)x_1(T), x_1(T) \rangle + o(\varepsilon).$$

We set

$$\begin{aligned} \bar{y}^\varepsilon(t) &= y^\varepsilon(t) - [\langle p(t), x_1(t) + x_2(t) \rangle + \frac{1}{2} \langle P(t)x_1(t), x_1(t) \rangle], \\ \bar{z}^\varepsilon(t) &= z^\varepsilon(t) - \{ \langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t) + \langle \sigma_x^T(t)p(t) + q(t), x_1(t) + x_2(t) \rangle \\ &\quad + \langle \delta\sigma_x^T(t)p(t) + \frac{1}{2}P(t)\delta\sigma(t) + \frac{1}{2}P^T(t)\delta\sigma(t), x_1(t) \rangle I_{E_\varepsilon}(t) \\ &\quad + \frac{1}{2} \langle [\sigma_{xx}^T(t)p(t) + P(t)\sigma_x(t) + \sigma_x^T(t)P(t) + Q(t)]x_1(t), x_1(t) \rangle \}. \end{aligned}$$

**Remark 3.2** It is important to note that the term  $\langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t)$  in the  $\bar{z}^\varepsilon(t)$  comes from the Itô's formula for  $\langle p(t), x_1(t) \rangle$ . Note that  $(\langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t))^i = (\langle p(t), \delta\sigma(t) \rangle)^i I_{E_\varepsilon}(t)$  for any integer  $i \geq 1$ , so it is not helpful to use the second-order Taylor expansion for treating this term, which is completely different from the terms  $x_1(t)$ ,  $x_2(t)$ ,  $x_1(t)I_{E_\varepsilon}(t)$  and  $x_1(t)(x_1(t))^T$ . In the following, we will show that this term is indeed in the variation of  $z$ , which is important for getting the variational equations for  $(y, z)$ .

Then by Lemma 3.1, we can get

$$\begin{aligned} \bar{y}^\varepsilon(t) &= \phi(\bar{x}(T)) + o(\varepsilon) + \int_t^T \{ [\langle p(s), \delta b(s) \rangle + \langle q(s), \delta\sigma(s) \rangle + \frac{1}{2} \langle P(s)\delta\sigma(s), \delta\sigma(s) \rangle] I_{E_\varepsilon}(s) \\ &\quad + f(s, x^\varepsilon(s), y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) - \langle f_x(s) + f_y(s)p(s) + f_z(s)[\sigma_x^T(s)p(s) + q(s)], x_1(s) + x_2(s) \rangle \\ &\quad - \frac{1}{2} \langle \{ f_y(s)P(s) + f_z(s)[\sigma_{xx}^T(s)p(s) + P(s)\sigma_x(s) + \sigma_x^T(s)P(s) + Q(s)] \\ &\quad + [I_{n \times n}, p(s), \sigma_x^T(s)p(s) + q(s)] D^2 f(s) [I_{n \times n}, p(s), \sigma_x^T(s)p(s) + q(s)]^T \} x_1(s), x_1(s) \rangle \} ds \\ &\quad - \int_t^T \bar{z}^\varepsilon(s) dW(s). \end{aligned} \tag{3.5}$$

**Remark 3.3** By the standard estimates of BSDEs, we can show that  $\bar{y}^\varepsilon(t) - \bar{y}(t) = O(\varepsilon)$ ,  $\bar{z}^\varepsilon(t) - \bar{z}(t) = O(\varepsilon)$  (see the proof of Theorem 3.4), which is the reason for constructing adjoint equations (3.1) and (3.2).

Suppose that

$$\begin{aligned} \bar{y}^\varepsilon(t) &= \bar{y}(t) + \hat{y}(t) + o(\varepsilon), \\ \bar{z}^\varepsilon(t) &= \bar{z}(t) + \hat{z}(t) + o(\varepsilon). \end{aligned} \tag{3.6}$$

Then from BSDE (3.5) and the Taylor expansion, we consider the following BSDE:

$$\begin{aligned} \hat{y}(t) &= \int_t^T \{ f_y(s)\hat{y}(s) + f_z(s)\hat{z}(s) + [\langle p(s), \delta b(s) \rangle + \langle q(s), \delta\sigma(s) \rangle + \frac{1}{2} \langle P(s)\delta\sigma(s), \delta\sigma(s) \rangle \\ &\quad + f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + \langle p(s), \delta\sigma(s) \rangle, u) - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s))] I_{E_\varepsilon}(s) \} ds \\ &\quad - \int_t^T \hat{z}(s) dW(s). \end{aligned} \tag{3.7}$$

In the following theorem, we will prove the above assumption (3.6).

**Theorem 3.4** Suppose (A1) and (A2) hold. Then, for any  $\beta \geq 2$ ,

$$E\left[\sup_{t \in [0, T]} |\bar{y}^\varepsilon(t) - \bar{y}(t)|^2 + \int_0^T |\bar{z}^\varepsilon(t) - \bar{z}(t)|^2 dt\right] = O(\varepsilon^2), \quad (3.8)$$

$$E\left[\sup_{t \in [0, T]} |\bar{y}^\varepsilon(t) - \bar{y}(t)|^\beta + \left(\int_0^T |\bar{z}^\varepsilon(t) - \bar{z}(t)|^2 dt\right)^{\beta/2}\right] = o(\varepsilon^{\beta/2}), \quad (3.9)$$

$$E\left[\sup_{t \in [0, T]} |\hat{y}(t)|^2 + \int_0^T |\hat{z}(t)|^2 dt\right] = O(\varepsilon^2), \quad (3.10)$$

$$E\left[\sup_{t \in [0, T]} |\bar{y}^\varepsilon(t) - \bar{y}(t) - \hat{y}(t)|^2 + \int_0^T |\bar{z}^\varepsilon(t) - \bar{z}(t) - \hat{z}(t)|^2 dt\right] = o(\varepsilon^2). \quad (3.11)$$

**Proof.** We first prove (3.8) and (3.9). Set

$$\begin{aligned} I_1(t) &= y^\varepsilon(t) - \bar{y}^\varepsilon(t), \quad I_2(t) = z^\varepsilon(t) - \bar{z}^\varepsilon(t) - \langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t), \\ I_3(t) &= \langle p(t), \delta b(t) \rangle + \langle q(t), \delta\sigma(t) \rangle + \frac{1}{2} \langle P(t) \delta\sigma(t), \delta\sigma(t) \rangle, \\ I_4(t) &= f(t, x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t)) - f(t, \bar{x}(t) + x_1(t) + x_2(t), \bar{y}^\varepsilon(t) + I_1(t), \bar{z}^\varepsilon(t) + I_2(t), \bar{u}(t)), \\ I_5(t) &= \delta\sigma_x^T(t) p(t) + \frac{1}{2} P(t) \delta\sigma(t) + \frac{1}{2} P^T(t) \delta\sigma(t), \\ I_6(t) &= [I_{n \times n}, p(t), \sigma_x^T(t) p(t) + q(t)]. \end{aligned}$$

Then we can get

$$\begin{aligned} \bar{y}^\varepsilon(t) - \bar{y}(t) &= o(\varepsilon) + \int_t^T \{I_3(s) I_{E_\varepsilon}(s) + I_4(s) + \tilde{f}_y(s)(\bar{y}^\varepsilon(s) - \bar{y}(s)) + \tilde{f}_z(s)(\bar{z}^\varepsilon(s) - \bar{z}(s)) \\ &\quad + [(x_1(s) + x_2(s))^T, I_1(s), I_2(s)] \tilde{D}^2 f(s) [(x_1(s) + x_2(s))^T, I_1(s), I_2(s)]^T \\ &\quad + f_z(s) \langle I_5(s), x_1(s) \rangle I_{E_\varepsilon}(s) - \frac{1}{2} \langle I_6(s) D^2 f(s) I_6^T(s) x_1(s), x_1(s) \rangle\} ds \\ &\quad - \int_t^T (\bar{z}^\varepsilon(s) - \bar{z}(s)) dW(s), \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} &f(t, \bar{x}(t) + x_1(t) + x_2(t), \bar{y}^\varepsilon(t) + I_1(t), \bar{z}^\varepsilon(t) + I_2(t), \bar{u}(t)) \\ &\quad - f(t, \bar{x}(t) + x_1(t) + x_2(t), \bar{y}(t) + I_1(t), \bar{z}(t) + I_2(t), \bar{u}(t)) \\ &= \tilde{f}_y(s)(\bar{y}^\varepsilon(s) - \bar{y}(s)) + \tilde{f}_z(s)(\bar{z}^\varepsilon(s) - \bar{z}(s)), \\ \tilde{f}_y(s) &= \int_0^1 f_y(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + I_1(s) + \mu(\bar{y}^\varepsilon(s) - \bar{y}(s)), \\ &\quad \bar{z}(s) + I_2(s) + \mu(\bar{z}^\varepsilon(s) - \bar{z}(s)), \bar{u}(s)) d\mu, \\ \tilde{f}_z(s) &= \int_0^1 f_z(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + I_1(s) + \mu(\bar{y}^\varepsilon(s) - \bar{y}(s)), \\ &\quad \bar{z}(s) + I_2(s) + \mu(\bar{z}^\varepsilon(s) - \bar{z}(s)), \bar{u}(s)) d\mu, \\ \tilde{D}^2 f(s) &= \int_0^1 \int_0^1 \lambda D^2 f(s, \bar{x}(s) + \lambda\mu(x_1(s) + x_2(s)), \bar{y}(s) + \lambda\mu I_1(s), \\ &\quad \bar{z}(s) + \lambda\mu I_2(s), \bar{u}(s)) d\lambda d\mu. \end{aligned}$$

Thus by Theorem 2.1, equation (3.12) and standard estimates of BSDEs, we can easily obtain (3.8) and (3.9). It is obviously for equation (3.10). Now we prove (3.11). Set

$$\tilde{x}^\varepsilon(t) = x^\varepsilon(t) - \bar{x}(t) - x_1(t) - x_2(t), \quad \tilde{y}^\varepsilon(t) = \bar{y}^\varepsilon(t) - \bar{y}(t) - \hat{y}(t), \quad \tilde{z}^\varepsilon(t) = \bar{z}^\varepsilon(t) - \bar{z}(t) - \hat{z}(t).$$

By equations (3.12) and (3.7), we can get

$$\begin{aligned} \tilde{y}^\varepsilon(t) = & o(\varepsilon) + \int_t^T \{ \tilde{f}_y(s) \tilde{y}^\varepsilon(s) + \tilde{f}_z(s) \tilde{z}^\varepsilon(s) + (\tilde{f}_y(s) - f_y(s)) \hat{y}(s) + (\tilde{f}_z(s) - f_z(s)) \hat{z}(s) \\ & + I_4(s) - [f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + \langle p(s), \delta\sigma(s) \rangle, u) - f(s)] I_{E_\varepsilon}(s) \\ & + [(x_1(s) + x_2(s))^T, I_1(s), I_2(s)] \tilde{D}^2 f(s) [(x_1(s) + x_2(s))^T, I_1(s), I_2(s)]^T \\ & + f_z(s) \langle I_5(s), x_1(s) \rangle I_{E_\varepsilon}(s) - \frac{1}{2} \langle I_6(s) D^2 f(s) I_6^T(s) x_1(s), x_1(s) \rangle \} ds \\ & - \int_t^T \tilde{z}^\varepsilon(s) dW(s). \end{aligned} \quad (3.13)$$

By Theorem 2.1, it is easy to check that we only need to show that

$$\begin{aligned} E[(\int_0^T |(\tilde{f}_y(s) - f_y(s)) \hat{y}(s) + (\tilde{f}_z(s) - f_z(s)) \hat{z}(s)| ds)^2] &= o(\varepsilon^2), \\ E[(\int_0^T |\langle I_6(s) (\tilde{D}^2 f(s) - \frac{1}{2} D^2 f(s)) I_6^T(s) x_1(s), x_1(s) \rangle| ds)^2] &= o(\varepsilon^2), \\ E[(\int_0^T |I_4(s) - [f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + \langle p(s), \delta\sigma(s) \rangle, u) - f(s)] I_{E_\varepsilon}(s)| ds)^2] &= o(\varepsilon^2). \end{aligned}$$

Note that

$$|\tilde{f}_y(s) - f_y(s)| + |\tilde{f}_z(s) - f_z(s)| \leq C(|x_1(s) + x_2(s)| + |I_1(s)| + |I_2(s)| + |\bar{y}^\varepsilon(s) - \bar{y}(s)| + |\bar{z}^\varepsilon(s) - \bar{z}(s)|),$$

and

$$\begin{aligned} E[(\int_0^T |q(s) x_1(s) \hat{z}(s)| ds)^2] &\leq E[\sup_{s \in [0, T]} |x_1(s)|^2 (\int_0^T |q(s)|^2 ds) (\int_0^T |\hat{z}(s)|^2 ds)] \\ &\leq (E[(\int_0^T |\hat{z}(s)|^2 ds)^2])^{1/2} (E[\sup_{s \in [0, T]} |x_1(s)|^8])^{1/4} (E[(\int_0^T |q(s)|^2 ds)^4])^{1/4} \\ &= o(\varepsilon^2), \end{aligned} \quad (3.14)$$

we can get  $E[(\int_0^T |(\tilde{f}_y(s) - f_y(s)) \hat{y}(s) + (\tilde{f}_z(s) - f_z(s)) \hat{z}(s)| ds)^2] = o(\varepsilon^2)$ . Since  $D^2 f$  is bounded, we can get that for each  $\beta \geq 2$ ,

$$E[(\int_0^T |(\tilde{D}^2 f(s) - \frac{1}{2} D^2 f(s))| |q(s)|^2 ds)^\beta] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Thus we can easily deduce  $E[(\int_0^T |\langle I_6(s) (\tilde{D}^2 f(s) - \frac{1}{2} D^2 f(s)) I_6^T(s) x_1(s), x_1(s) \rangle| ds)^2] = o(\varepsilon^2)$ . It is easy to verify that

$$\begin{aligned} &|I_4(s) - [f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + \langle p(s), \delta\sigma(s) \rangle, u) - f(s)] I_{E_\varepsilon}(s)| \\ &\leq C\{|\tilde{x}^\varepsilon(s)| + [|x_1(s) + x_2(s)| + |\bar{y}^\varepsilon(s) - \bar{y}(s)| + |\bar{z}^\varepsilon(s) - \bar{z}(s)| + |I_1(s)| + |I_2(s)|] I_{E_\varepsilon}(s)\}. \end{aligned}$$



Since

$$\begin{aligned}
E[(\int_0^T |q(s)x_1(s)|I_{E_\varepsilon}(s)ds)^2] &\leq E[\sup_{s \in [0, T]} |x_1(s)|^2 \int_{E_\varepsilon} |q(s)|^2 ds] \varepsilon \\
&\leq (E[\sup_{s \in [0, T]} |x_1(s)|^4])^{1/2} (E[(\int_{E_\varepsilon} |q(s)|^2 ds)^2])^{1/2} \varepsilon \\
&= o(\varepsilon^2),
\end{aligned}$$

we can obtain  $E[(\int_0^T |I_4(s) - [f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + \langle p(s), \delta\sigma(s) \rangle, u) - f(s)]I_{E_\varepsilon}(s)|ds)^2] = o(\varepsilon^2)$ . The proof is complete.  $\square$

Thus we obtain the following variational equation for BSDE (1.3):

$$\begin{aligned}
y^\varepsilon(t) &= \bar{y}(t) + \langle p(t), x_1(t) + x_2(t) \rangle + \frac{1}{2} \langle P(t)x_1(t), x_1(t) \rangle + \hat{y}(t) + o(\varepsilon), \\
z^\varepsilon(t) &= \bar{z}(t) + \langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t) + \langle \sigma_x^T(t)p(t) + q(t), x_1(t) + x_2(t) \rangle \\
&\quad + \langle \delta\sigma_x^T(t)p(t) + \frac{1}{2}P(t)\delta\sigma(t) + \frac{1}{2}P^T(t)\delta\sigma(t), x_1(t) \rangle I_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2} \langle [\sigma_{xx}^T(t)p(t) + P(t)\sigma_x(t) + \sigma_x^T(t)P(t) + Q(t)]x_1(t), x_1(t) \rangle + \hat{z}(t) + o(\varepsilon).
\end{aligned} \tag{3.15}$$

**Remark 3.5** We can also give the variational equations for BSDE (1.3) as in [12]. Set

$$y_1(t) = \langle p(t), x_1(t) \rangle, \quad z_1(t) = \langle p(t), \delta\sigma(t) \rangle I_{E_\varepsilon}(t) + \langle \sigma_x^T(t)p(t) + q(t), x_1(t) \rangle, \tag{3.16}$$

it is easy to check that  $(y_1, z_1)$  satisfies the following BSDE:

$$\begin{cases} -dy_1(t) = \{ \langle f_x(t), x_1(t) \rangle + f_y(t)y_1(t) + f_z(t)z_1(t) \\ \quad - [f_z(t)\langle p(t), \delta\sigma(t) \rangle + \langle q(t), \delta\sigma(t) \rangle] I_{E_\varepsilon}(t) \} dt - z_1(t)dW(t), \\ y_1(T) = \langle \phi_x(\bar{x}(T)), x_1(T) \rangle. \end{cases} \tag{3.17}$$

Set

$$\begin{aligned}
y_2(t) &= \langle p(t), x_2(t) \rangle + \frac{1}{2} \langle P(t)x_1(t), x_1(t) \rangle + \hat{y}(t), \\
z_2(t) &= \langle \sigma_x^T(t)p(t) + q(t), x_2(t) \rangle + \langle \delta\sigma_x^T(t)p(t) + \frac{1}{2}P(t)\delta\sigma(t) + \frac{1}{2}P^T(t)\delta\sigma(t), x_1(t) \rangle I_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2} \langle [\sigma_{xx}^T(t)p(t) + P(t)\sigma_x(t) + \sigma_x^T(t)P(t) + Q(t)]x_1(t), x_1(t) \rangle + \hat{z}(t),
\end{aligned} \tag{3.18}$$

it is easy to verify that

$$\begin{cases} -dy_2(t) = \{ \langle f_x(t), x_2(t) \rangle + f_y(t)y_2(t) + f_z(t)z_2(t) \\ \quad + \frac{1}{2}[(x_1(t))^T, y_1(t), z_1(t)]D^2f(t)[(x_1(t))^T, y_1(t), z_1(t)]^T \\ \quad + [\langle q(t), \delta\sigma(t) \rangle - \frac{1}{2}f_{zz}(t)(\langle p(t), \delta\sigma(t) \rangle)^2 + f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t) + \langle p(t), \delta\sigma(t) \rangle, u) \\ \quad - f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t))] I_{E_\varepsilon}(t) + \langle L(t), x_1(t) \rangle I_{E_\varepsilon}(t) \} dt - z_2(t)dW(t), \\ y_2(T) = \langle \phi_x(\bar{x}(T)), x_2(T) \rangle + \frac{1}{2} \langle \phi_{xx}(\bar{x}(T))x_1(T), x_1(T) \rangle, \end{cases} \tag{3.19}$$

where  $\langle L(t), x_1(t) \rangle_{I_{E_\varepsilon}}(t) = o(\varepsilon)$ , so we do not give the explicit formula for  $L(t)$ . Here we use

$$[f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t) + \langle p(t), \delta\sigma(t) \rangle, u) - f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t))]I_{E_\varepsilon}(t)$$

to completely deal with the term  $\langle p(t), \delta\sigma(t) \rangle_{I_{E_\varepsilon}}(t)$  in the variation of  $z$ , so the terms  $f_z(t)\langle p(t), \delta\sigma(t) \rangle$  and  $\frac{1}{2}f_{zz}(t)(\langle p(t), \delta\sigma(t) \rangle)^2$  are repeated in  $f_z(t)z_1(t)$  and  $\frac{1}{2}f_{zz}(t)(z_1(t))^2$ . Noting equations (3.16) and (3.18), then the adjoint equations for  $(z_1(t))^2$  and other terms are essentially for  $x_1(t)$ ,  $x_2(t)$  and  $x_1(t)(x_1(t))^T$ , which is solved in [12]. In order to further explain the difference of expansions for SDE and BSDE, we consider the following equations:

$$\begin{cases} -d\tilde{y}_1(t) = \{ \langle f_x(t), x_1(t) \rangle + f_y(t)\tilde{y}_1(t) + f_z(t)\tilde{z}_1(t) \} dt - \tilde{z}_1(t)dW(t), \\ \tilde{y}_1(T) = \langle \phi_x(\bar{x}(T)), x_1(T) \rangle, \end{cases} \quad (3.20)$$

$$\begin{cases} -d\tilde{y}_2(t) = \{ \langle f_x(t), x_2(t) \rangle + f_y(t)\tilde{y}_2(t) + f_z(t)\tilde{z}_2(t) \\ \quad + \frac{1}{2}[(x_1(t))^T, \tilde{y}_1(t), \tilde{z}_1(t)]D^2f(t)[(x_1(t))^T, \tilde{y}_1(t), \tilde{z}_1(t)]^T \\ \quad + [f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t) + \langle p(t), \delta\sigma(t) \rangle, u) - f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)) \\ \quad - f_z(t)\langle p(t), \delta\sigma(t) \rangle - \frac{1}{2}f_{zz}(t)(\langle p(t), \delta\sigma(t) \rangle)^2]I_{E_\varepsilon}(t) \} dt - \tilde{z}_2(t)dW(t), \\ \tilde{y}_2(T) = \langle \phi_x(\bar{x}(T)), x_2(T) \rangle + \frac{1}{2}\langle \phi_{xx}(\bar{x}(T))x_1(T), x_1(T) \rangle. \end{cases} \quad (3.21)$$

By the standard estimates of BSDEs, it is easy to show that

$$E[\sup_{t \in [0, T]} |y_1(t) + y_2(t) - \tilde{y}_1(t) - \tilde{y}_2(t)|^2 + \int_0^T |z_1(t) + z_2(t) - \tilde{z}_1(t) - \tilde{z}_2(t)|^2 dt] = o(\varepsilon^2).$$

Thus by equation (3.15), we can get

$$y^\varepsilon(t) = \bar{y}(t) + \tilde{y}_1(t) + \tilde{y}_2(t) + o(\varepsilon),$$

$$z^\varepsilon(t) = \bar{z}(t) + \tilde{z}_1(t) + \tilde{z}_2(t) + o(\varepsilon).$$

The main difference is equation (3.21) which is due to the term  $\langle p(t), \delta\sigma(t) \rangle_{I_{E_\varepsilon}}(t)$  in the variation of  $z$ . If  $f$  is independent of  $z$ , the variational equations for  $(y, z)$  are the same as in [12], which is pointed in [15].

Now we consider the maximum principle. From equation (3.15), we get

$$J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) = y^\varepsilon(0) - \bar{y}(0) = \hat{y}(0) + o(\varepsilon).$$

Define the following adjoint equation for BSDE (3.7):

$$\begin{cases} d\gamma(t) = f_y(t)\gamma(t)dt + f_z(t)\gamma(t)dW(t), \\ \gamma(0) = 1. \end{cases}$$

Applying Itô's formula to  $\gamma(t)\hat{y}(t)$ , we can obtain

$$\begin{aligned}\hat{y}(0) = & E[\int_0^T \gamma(s)[\langle p(s), \delta b(s) \rangle + \langle q(s), \delta \sigma(s) \rangle + \frac{1}{2} \langle P(s) \delta \sigma(s), \delta \sigma(s) \rangle \\ & + f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + \langle p(s), \delta \sigma(s) \rangle, u) - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s))] I_{E_\varepsilon}(s) ds].\end{aligned}\quad (3.22)$$

Note that  $\gamma(s) > 0$ , then we define the following function:

$$\begin{aligned}\mathcal{H}(t, x, y, z, u, p, q, P) = & \langle p, b(t, x, u) \rangle + \langle q, \sigma(t, x, u) \rangle \\ & + \frac{1}{2} \langle P(\sigma(t, x, u) - \sigma(t, \bar{x}, \bar{u})), \sigma(t, x, u) - \sigma(t, \bar{x}, \bar{u}) \rangle \\ & + f(t, x, y, z + \langle p, \sigma(t, x, u) - \sigma(t, \bar{x}, \bar{u}) \rangle, u),\end{aligned}\quad (3.23)$$

where  $(p, q, P)$  is defined in equations (3.1) and (3.2). Thus we obtain the following maximum principle.

**Theorem 3.6** *Suppose (A1) and (A2) hold. Let  $\bar{u}(\cdot)$  be an optimal control and  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$  be the corresponding solution. Then*

$$\mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), u, p(t), q(t), P(t)) \geq \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t), p(t), q(t), P(t)), \forall u \in U, a.e., a.s., \quad (3.24)$$

where  $\mathcal{H}(\cdot)$  is defined in (3.23).

If the control domain  $U$  is convex, we can get the following corollary which is obtained by Peng in [13].

**Corollary 3.7** *Let the assumptions as in Theorem 3.6. If  $U$  is convex and  $b, \sigma, f$  are continuously differentiable with respect to  $u$ , then*

$$\langle b_u^T(t)p(t) + \sigma_u^T(t)q(t) + f_z(t)\sigma_u^T(t)p(t) + f_u(t), u - \bar{u}(t) \rangle \geq 0, \quad \forall u \in U, a.e., a.s..$$

Now we give an example to compare our result with the result in [20, 24].

**Example 3.8** *Suppose  $n = d = k = 1$ .  $U$  is a given subset in  $\mathbb{R}$ . Consider the following control system:*

$$\begin{aligned}dx(t) &= u(t)dW(t), \quad x(0) = 0, \\ y(t) &= x(T) + \int_t^T f(z(s))ds - \int_t^T z(s)dW(s).\end{aligned}$$

In this case, our maximum principle is

$$f(\bar{z}(t) + u - \bar{u}(t)) - f(\bar{z}(t)) \geq 0, \quad \forall u \in U, a.e., a.s.. \quad (3.25)$$

Note that

$$y(t) - \int_0^t u(s)dW(s) = \int_t^T f(z(s) - u(s) + u(s))ds - \int_t^T (z(s) - u(s))dW(s),$$

then by comparison theorem of BSDE, it is easy to check that inequality (3.25) is a sufficient condition. For the case  $U = \{0, 1\}$ ,  $f(0) = 0$ ,  $f'(0) < 0$ ,  $f(1) > 0$ ,  $f(-1) < 0$ , it is easy to verify that  $(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = (0, 0, 0, 0)$  satisfies (3.25), thus  $\bar{u} = 0$  is an optimal control. But  $f_z(\bar{z}(t))(1 - \bar{u}(t)) < 0$ , which implies that  $\bar{u} = 0$  is not an optimal control for the case  $U = [0, 1]$ . The maximum principle in [24] is  $f_z(\bar{z}(t))(u - \bar{u}(t)) \geq 0$ ,  $\forall u \in U$ , a.e., a.s., which only cover the case  $U$  is convex. The maximum principle in [20] contains two unknown parameters.

**Remark 3.9** In [20, 24], the authors consider the control system which consists of SDE (1.1) and the following state equation:

$$y(t) = y_0 - \int_0^t f(s, x(s), y(s), v(s), u(s))ds + \int_0^t v(s)dW(s), \quad (3.26)$$

where the set of all admissible controls

$$\tilde{\mathcal{U}}[0, T] = \{(u, y_0, v) \in \mathcal{U}[0, T] \times \mathbb{R} \times M^2(0, T) : y(T) = \phi(x(T))\}.$$

The optimal control problem is to minimize  $J(u(\cdot), y_0, v(\cdot)) = y_0$  over  $\tilde{\mathcal{U}}[0, T]$ . Obviously, this problem is equivalent to Peng's problem. Thus our maximum principle also completely solves this control problem.

### 3.2 Multi-dimensional case

In this subsection, we extend Peng's problem to multi-dimensional case, i.e., the functions in BSDE (1.3) are  $m$ -dimensional,  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ . The cost functional is defined by

$$J(u(\cdot)) = h(y(0)), \quad (3.27)$$

where  $h : \mathbb{R}^m \rightarrow \mathbb{R}$ . For deriving the variational equation for BSDE (1.3), we use the following notation.

$$\begin{aligned} W(t) &= (W^1(t), \dots, W^d(t))^T, \quad \phi(x) = (\phi^1(x), \dots, \phi^m(x))^T, \\ \sigma(t, x, u) &= (\sigma^{ij}(t, x, u)), \quad i = 1, \dots, n, j = 1, \dots, d, \\ \sigma^j(t, x, u) &= (\sigma^{1j}(t, x, u), \dots, \sigma^{nj}(t, x, u))^T, \quad j = 1, \dots, d, \\ f(t, x, y, z, u) &= (f^1(t, x, y, z, u), \dots, f^m(t, x, y, z, u))^T, \\ y(t) &= (y^1(t), \dots, y^m(t))^T, \quad z(t) = (z^{ij}(t)), i \leq m, j \leq d, \\ z^j(t) &= (z^{1j}(t), \dots, z^{mj}(t))^T, \quad j = 1, \dots, d. \end{aligned} \quad (3.28)$$

We introduce the following adjoint equations: for  $i = 1, \dots, m$ ,

$$\begin{cases} -dp_i(t) = F_i(t)dt - \sum_{j=1}^d q_i^j(t)dW^j(t), \\ p_i(T) = \phi_x^i(\bar{x}(T)), \end{cases} \quad (3.29)$$

$$\begin{cases} -dP_i(t) = G_i(t)dt - \sum_{j=1}^d Q_i^j(t)dW^j(t), \\ P_i(T) = \phi_{xx}^i(\bar{x}(T)), \end{cases} \quad (3.30)$$

where  $F_i(t)$  and  $G_i(t)$  is given after the following notations:

$$\begin{aligned} p(t) &= [p_1(t), \dots, p_m(t)]_{n \times m}, \quad q^j(t) = [q_1^j(t), \dots, q_m^j(t)]_{n \times m}, \\ p_l(t) &= (p_l^1(t), \dots, p_l^n(t))^T, \quad q_l^j(t) = (q_l^{1j}(t), \dots, q_l^{mj}(t))^T, \\ b_{xx}^T(t)p_l(t) &= \sum_{i=1}^n p_l^i(t)(b_{xx}^i(t))^T, \quad (\sigma_{xx}^j(t))^T p_l(t) = \sum_{i=1}^n p_l^i(t)(\sigma_{xx}^{ij}(t))^T, \\ (\sigma_{xx}^j(t))^T q_l^j(t) &= \sum_{i=1}^n q_l^{ij}(t)(\sigma_{xx}^{ij}(t))^T, \quad l = 1, \dots, m, \quad j = 1, \dots, d. \end{aligned} \quad (3.31)$$

$$\begin{aligned}
F_i(t) &= b_x^T(t)p_i(t) + f_x^i(t) + \sum_{l=1}^m f_{y^l}^i(t)p_l(t) + \sum_{j=1}^d (\sigma_x^j(t))^T q_i^j(t) \\
&\quad + \sum_{j=1}^d \sum_{l=1}^m f_{z^{lj}}^i(t)[(\sigma_x^j(t))^T p_l(t) + q_l^j(t)], \\
G_i(t) &= P_i(t)b_x(t) + (b_x(t))^T P_i(t) + \sum_{l=1}^m f_{y^l}^i(t)P_l(t) + \sum_{j=1}^d [Q_i^j(t)\sigma_x^j(t) + (\sigma_x^j(t))^T Q_i^j(t) \\
&\quad + (\sigma_{xx}^j(t))^T q_i^j(t) + (\sigma_x^j(t))^T P_i(t)\sigma_x^j(t)] + \sum_{j=1}^d \sum_{l=1}^m [f_{z^{lj}}^i(t)P_l(t)\sigma_x^j(t) + f_{z^{lj}}^i(t)(\sigma_x^j(t))^T P_l(t) \\
&\quad + f_{z^{lj}}^i(t)Q_l^j(t) + f_{z^{lj}}^i(t)(\sigma_{xx}^j(t))^T p_l(t)] + b_{xx}^T(t)p_i(t) + [I_{n \times n}, p(t), (\sigma_x^1(t))^T p(t) + q^1(t), \dots, \\
&\quad (\sigma_x^d(t))^T p(t) + q^d(t)] D^2 f^i(t) [I_{n \times n}, p(t), (\sigma_x^1(t))^T p(t) + q^1(t), \dots, (\sigma_x^d(t))^T p(t) + q^d(t)]^T,
\end{aligned} \tag{3.32}$$

where  $D^2 f^i$  is the Hessian matrix of  $f^i$  with respect to  $(x, y, z^1, \dots, z^d)$ . Let  $\hat{y}(t) = (\hat{y}^1(t), \dots, \hat{y}^m(t))^T$ ,  $\hat{z}(t) = (\hat{z}^{ij}(t))$  be the solution of the following BSDE:

$$\begin{aligned}
\hat{y}(t) &= \int_t^T [f_y(s)\hat{y}(s) + \sum_{j=1}^d f_{z^j}(s)\hat{z}^j(s) + \{p^T(s)\delta b(s) + \sum_{j=1}^d [(q^j(s))^T \delta \sigma^j(s) + \frac{1}{2}P^T(s)\delta \sigma^j(s)\delta \sigma^j(s)] \\
&\quad + f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + p^T(s)\delta \sigma(s), u) - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s))\} I_{E_\varepsilon}(s)] ds \\
&\quad - \sum_{j=1}^d \int_t^T \hat{z}^j(s) dW^j(s),
\end{aligned} \tag{3.33}$$

where

$$P(t) = [P_1(t), \dots, P_m(t)], \quad P^T(s)\delta \sigma^j(s)\delta \sigma^j(s) = (\langle P_1(s)\delta \sigma^j(s), \delta \sigma^j(s) \rangle, \dots, \langle P_m(s)\delta \sigma^j(s), \delta \sigma^j(s) \rangle)^T.$$

Similar to the analysis in Theorem 3.4, we can get the following variational principle:

$$\begin{aligned}
y^{i;\varepsilon}(t) &= \bar{y}^i(t) + \langle p_i(t), x_1(t) + x_2(t) \rangle + \frac{1}{2} \langle P_i(t)x_1(t), x_1(t) \rangle + \hat{y}^i(t) + o(\varepsilon), \\
z^{ij;\varepsilon}(t) &= \bar{z}^{ij}(t) + \langle p_i(t), \delta \sigma^j(t) \rangle I_{E_\varepsilon}(t) + \langle (\sigma_x^j(t))^T p_i(t) + q_i^j(t), x_1(t) + x_2(t) \rangle \\
&\quad + \langle (\delta \sigma_x^j(t))^T p_i(t) + \frac{1}{2}P_i(t)\delta \sigma^j(t) + \frac{1}{2}P_i^T(t)\delta \sigma^j(t), x_1(t) \rangle I_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2} \langle [(\sigma_{xx}^j(t))^T p_i(t) + P_i(t)\sigma_x^j(t) + (\sigma_x^j(t))^T P_i(t) + Q_i^j(t)]x_1(t), x_1(t) \rangle \\
&\quad + \hat{z}^{ij}(t) + o(\varepsilon), \quad i = 1, \dots, m, \quad j = 1, \dots, d.
\end{aligned} \tag{3.34}$$

Let  $h \in C^1(\mathbb{R}^m)$ . Then we get

$$J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) = \langle h_y(\bar{y}(0)), \hat{y}(0) \rangle + o(\varepsilon).$$

We introduce the following adjoint equation for BSDE (3.33).

$$\begin{cases} d\gamma(t) = f_y^T(t)\gamma(t)dt + \sum_{j=1}^d f_{z^j}^T(t)\gamma(t)dW^j(t), \\ \gamma(0) = h_y(\bar{y}(0)). \end{cases} \tag{3.35}$$

Applying Itô's formula to  $\langle \gamma(t), \hat{y}(t) \rangle$ , we can get the following maximum principle.

**Theorem 3.10** Suppose (A1) and (A2) hold. Let  $\bar{u}(\cdot)$  be an optimal control and  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$  be the corresponding solution. The cost function is defined in (3.27) and  $h \in C^1(\mathbb{R}^m)$ . Then

$$\begin{aligned} & \langle \gamma(t), p^T(t) \delta b(t) + \sum_{j=1}^d [(q^j(t))^T \delta \sigma^j(t) + \frac{1}{2} P^T(t) \delta \sigma^j(t) \delta \sigma^j(t)] \\ & + f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t) + p^T(t) \delta \sigma(t), u) - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s)) \rangle \\ & \geq 0, \quad \forall u \in U, \text{ a.e., a.s.,} \end{aligned} \quad (3.36)$$

where  $p, q^j, P, \gamma$  are given in equations (3.29), (3.30), (3.32) and (3.35).

## 4 Problem with state constraint

For the simplicity of presentation, suppose  $d = m = 1$ , the multi-dimensional case can be treated with the same method.

We consider the control system: SDE (1.1) and BSDE (1.3). The cost function  $J(u(\cdot))$  is defined in (1.4). In addition, we consider the following state constraint:

$$E[\varphi(x(T), y(0))] = 0, \quad (4.1)$$

where  $\varphi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ . We need the following assumption:

**(A3)**  $\varphi$  is twice continuously differentiable with respect to  $(x, y)$ ;  $D^2\varphi$  is bounded;  $D\varphi$  is bounded by  $C(1 + |x| + |y|)$ .

Define all admissible controls as follows:

$$\mathcal{U}_{ad}[0, T] = \{u(\cdot) \in \mathcal{U}[0, T] : E[\varphi(x(T), y(0))] = 0\}.$$

The control problem is to minimize  $J(u(\cdot))$  over  $\mathcal{U}_{ad}[0, T]$ .

Let  $\bar{u}(\cdot) \in \mathcal{U}_{ad}[0, T]$  be an optimal control and  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$  be the corresponding solution of equations (1.1) and (1.3). Similarly, we define  $(x(\cdot), y(\cdot), z(\cdot), u(\cdot))$  for any  $u(\cdot) \in \mathcal{U}[0, T]$ . For any  $\rho > 0$ , define the following cost functional on  $\mathcal{U}[0, T]$ :

$$J_\rho(u(\cdot)) = \{[(y(0) - \bar{y}(0)) + \rho]^2 + |E[\varphi(x(T), y(0))]|^2\}^{1/2}. \quad (4.2)$$

It is easy to check that

$$\begin{cases} J_\rho(u(\cdot)) > 0, \quad \forall u(\cdot) \in \mathcal{U}[0, T], \\ J_\rho(\bar{u}(\cdot)) = \rho \leq \inf_{u \in \mathcal{U}[0, T]} J_\rho(u(\cdot)) + \rho. \end{cases}$$

In order to use well-known Ekeland's variational principle, we define the following metric on  $\mathcal{U}[0, T]$ :

$$d(u(\cdot), v(\cdot)) = E\left[\int_0^T I_{\{u \neq v\}}(t, \omega) dt\right].$$

Suppose that  $(\mathcal{U}[0, T], d)$  is a complete space and  $J_\rho(\cdot)$  is continuous, otherwise we can use the technique in [18, 20] and the result is the same. Thus, by Ekeland's variational principle, there exists a  $u_\rho(\cdot) \in \mathcal{U}[0, T]$  such that

$$J_\rho(u_\rho(\cdot)) \leq \rho, \quad d(u_\rho(\cdot), \bar{u}(\cdot)) \leq \sqrt{\rho}, \quad (4.3)$$

$$J_\rho(u(\cdot)) - J_\rho(u_\rho(\cdot)) + \sqrt{\rho}d(u_\rho(\cdot), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in \mathcal{U}[0, T].$$

For any  $\varepsilon > 0$ , let  $E_\varepsilon \subset [0, T]$  with  $|E_\varepsilon| = \varepsilon$ , define

$$u_\rho^\varepsilon(t) = u_\rho(t)I_{E_\varepsilon^c}(t) + uI_{E_\varepsilon}(t), \quad \forall u \in U.$$

It is easy to check that  $d(u_\rho(\cdot), u_\rho^\varepsilon(\cdot)) \leq \varepsilon$ . Let  $(x_\rho(\cdot), y_\rho(\cdot), z_\rho(\cdot))$  be the solution corresponding to  $u_\rho(\cdot)$ . Similarly for  $(x_\rho^\varepsilon(\cdot), y_\rho^\varepsilon(\cdot), z_\rho^\varepsilon(\cdot), u_\rho^\varepsilon(\cdot))$ . Thus by (4.3), we can get

$$\begin{aligned} 0 &\leq J_\rho(u_\rho^\varepsilon(\cdot)) - J_\rho(u_\rho(\cdot)) + \sqrt{\rho}\varepsilon \\ &\leq \lambda_\rho[y_\rho^\varepsilon(0) - y_\rho(0)] + \mu_\rho\{E[\varphi(x_\rho^\varepsilon(T), y_\rho^\varepsilon(0))] - E[\varphi(x_\rho(T), y_\rho(0))]\} + \sqrt{\rho}\varepsilon + o(\varepsilon), \end{aligned} \quad (4.4)$$

where

$$\lambda_\rho = J_\rho(u_\rho(\cdot))^{-1}[(y_\rho(0) - \bar{y}(0)) + \rho], \quad \mu_\rho = J_\rho(u_\rho(\cdot))^{-1}E[\varphi(x_\rho(T), y_\rho(0))].$$

The same analysis as in Theorem 3.4, let  $(p^\rho(\cdot), q^\rho(\cdot))$  and  $(P^\rho(\cdot), Q^\rho(\cdot))$  be respectively the solutions of equations (3.1) and (3.2) with  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{u}(\cdot))$  replaced by  $(x_\rho(\cdot), y_\rho(\cdot), z_\rho(\cdot), u_\rho(\cdot))$ , and let all the coefficients be added by a superscript  $\rho$ . Then

$$y_\rho^\varepsilon(0) - y_\rho(0) = \hat{y}_\rho(0) + o(\varepsilon), \quad (4.5)$$

where

$$\begin{aligned} \hat{y}_\rho(t) = & \int_t^T \{f_y^\rho(s)\hat{y}_\rho(s) + f_z^\rho(s)\hat{z}_\rho(s) + [\langle p^\rho(s), \delta b^\rho(s) \rangle + \langle q^\rho(s), \delta \sigma^\rho(s) \rangle + \frac{1}{2}\langle P^\rho(s)\delta \sigma^\rho(s), \delta \sigma^\rho(s) \rangle \\ & + f(s, x_\rho(s), y_\rho(s), z_\rho(s) + \langle p^\rho(s), \delta \sigma^\rho(s) \rangle, u) - f(s, x_\rho(s), y_\rho(s), z_\rho(s), u_\rho(s))]I_{E_\varepsilon}(s)\}ds \\ & - \int_t^T \hat{z}_\rho(s)dW(s). \end{aligned} \quad (4.6)$$

Similarly, let

$$\begin{cases} -dp_0^\rho(t) = [(b_x^\rho(t))^T p_0^\rho(t) + (\sigma_x^\rho(t))^T q_0^\rho(t)]dt - q_0^\rho(t)dW(t), \\ p_0(T) = \mu_\rho \varphi_x(x_\rho(T), y_\rho(0)), \end{cases}$$

$$\begin{cases} -dP_0^\rho(t) = [(b_x^\rho(t))^T P_0^\rho(t) + P_0^\rho(t)b_x^\rho(t) + (\sigma_x^\rho(t))^T P_0^\rho(t)\sigma_x^\rho(t) + (\sigma_x^\rho(t))^T Q_0^\rho(t) + Q_0^\rho(t)\sigma_x^\rho(t) \\ + (b_{xx}^\rho(t))^T p_0^\rho(t) + (\sigma_{xx}^\rho(t))^T q_0^\rho(t)]dt - Q_0^\rho(t)dW(t), \\ P_0(T) = \mu_\rho \varphi_{xx}(x_\rho(T), y_\rho(0)). \end{cases}$$

Then

$$\begin{aligned} &\mu_\rho\{E[\varphi(x_\rho^\varepsilon(T), y_\rho^\varepsilon(0))] - E[\varphi(x_\rho(T), y_\rho(0))]\} \\ &= E[\int_0^T \{\langle p_0^\rho(s), \delta b^\rho(s) \rangle + \langle q_0^\rho(s), \delta \sigma^\rho(s) \rangle + \frac{1}{2}\langle P_0^\rho(s)\delta \sigma^\rho(s), \delta \sigma^\rho(s) \rangle\}I_{E_\varepsilon}(s)ds] \\ &\quad + \mu_\rho E[\varphi_y(x_\rho(T), y_\rho(0))]\hat{y}_\rho(0) + o(\varepsilon). \end{aligned} \quad (4.7)$$

Define the following adjoint equation for BSDE (4.6):

$$\begin{cases} d\gamma^\rho(t) = f_y^\rho(t)\gamma^\rho(t)dt + f_z^\rho(t)\gamma^\rho(t)dW(t), \\ \gamma^\rho(0) = \lambda_\rho + \mu_\rho E[\varphi_y(x_\rho(T), y_\rho(0))]. \end{cases}$$

Then we can get

$$\begin{aligned} & \{\lambda_\rho + \mu_\rho E[\varphi_y(x_\rho(T), y_\rho(0))]\}\hat{y}_\rho(0) \\ &= E[\int_0^T \gamma^\rho(s)\{\langle p^\rho(s), \delta b^\rho(s) \rangle + \langle q^\rho(s), \delta \sigma^\rho(s) \rangle + \frac{1}{2}\langle P^\rho(s)\delta \sigma^\rho(s), \delta \sigma^\rho(s) \rangle \\ & \quad + f(s, x_\rho(s), y_\rho(s), z_\rho(s) + \langle p^\rho(s), \delta \sigma^\rho(s) \rangle, u) - f(s, x_\rho(s), y_\rho(s), z_\rho(s), u_\rho(s))\}I_{E_\varepsilon}(s)ds]. \end{aligned} \quad (4.8)$$

Define the following function:

$$\begin{aligned} & \mathcal{H}(t, x, y, z, u, x', u', p_0, q_0, P_0, p, q, P, \gamma) \\ &= \langle p_0 + \gamma p, b(t, x, u) \rangle + \langle q_0 + \gamma q, \sigma(t, x, u) \rangle \\ & \quad + \frac{1}{2}\langle (P_0 + \gamma P)(\sigma(t, x, u) - \sigma(t, x', u')), \sigma(t, x, u) - \sigma(t, x', u') \rangle \\ & \quad + \gamma(t)f(t, x, y, z + \langle p, \sigma(t, x, u) - \sigma(t, x', u') \rangle, u). \end{aligned} \quad (4.9)$$

It follows from (4.4), (4.5), (4.7) and (4.8) that

$$\begin{aligned} 0 &\leq E[\int_0^T \{\mathcal{H}(t, x_\rho(t), y_\rho(t), z_\rho(t), u, x_\rho(t), u_\rho(t), p_0^\rho(t), q_0^\rho(t), P_0^\rho(t), p^\rho(t), q^\rho(t), P^\rho(t), \gamma^\rho(t)) \\ & \quad - \mathcal{H}(t, x_\rho(t), y_\rho(t), z_\rho(t), u_\rho(t), x_\rho(t), u_\rho(t), p_0^\rho(t), q_0^\rho(t), P_0^\rho(t), p^\rho(t), q^\rho(t), P^\rho(t), \gamma^\rho(t))\}I_{E_\varepsilon}(t)dt] \\ & \quad + \sqrt{\rho}\varepsilon + o(\varepsilon). \end{aligned}$$

Thus we obtain

$$\begin{aligned} & \mathcal{H}(t, x_\rho(t), y_\rho(t), z_\rho(t), u, x_\rho(t), u_\rho(t), p_0^\rho(t), q_0^\rho(t), P_0^\rho(t), p^\rho(t), q^\rho(t), P^\rho(t), \gamma^\rho(t)) \\ & \geq \mathcal{H}(t, x_\rho(t), y_\rho(t), z_\rho(t), u_\rho(t), x_\rho(t), u_\rho(t), p_0^\rho(t), q_0^\rho(t), P_0^\rho(t), p^\rho(t), q^\rho(t), P^\rho(t), \gamma^\rho(t)) \\ & \quad - \sqrt{\rho}, \quad \forall u \in U, \quad \text{a.e., a.s..} \end{aligned}$$

Obviously,  $|\lambda_\rho|^2 + |\mu_\rho|^2 = 1$ . Thus there exists a subsequence of  $(\lambda_\rho, \mu_\rho)$  which converges to  $(\lambda, \mu)$  with  $|\lambda|^2 + |\mu|^2 = 1$  as  $\rho \rightarrow 0$ . Note that  $d(u_\rho(\cdot), \bar{u}(\cdot)) \leq \sqrt{\rho}$ , then we can get for further subsequence

$$\begin{aligned} & (x_\rho(\cdot), y_\rho(\cdot), z_\rho(\cdot), u_\rho(\cdot), p_0^\rho(\cdot), q_0^\rho(\cdot), P_0^\rho(\cdot), p^\rho(\cdot), q^\rho(\cdot), P^\rho(\cdot), \gamma^\rho(\cdot)) \rightarrow \\ & (\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{u}(\cdot), p_0(\cdot), q_0(\cdot), P_0(\cdot), p(\cdot), q(\cdot), P(\cdot), \gamma(\cdot)), \quad \text{a.e., a.s.,} \end{aligned}$$

where  $(p(\cdot), q(\cdot))$  and  $(P(\cdot), Q(\cdot))$  are respectively the solutions of equations (3.1) and (3.2),

$$\begin{cases} -dp_0(t) = [(b_x(t))^T p_0(t) + (\sigma_x(t))^T q_0(t)]dt - q_0(t)dW(t), \\ p_0(T) = \mu\varphi_x(\bar{x}(T), \bar{y}(0)), \end{cases} \quad (4.10)$$



$$\left\{ \begin{array}{l} -dP_0(t) = [(b_x(t))^T P_0(t) + P_0(t)b_x(t) + (\sigma_x(t))^T P_0(t)\sigma_x(t) + (\sigma_x(t))^T Q_0(t) + Q_0(t)\sigma_x(t) \\ \quad + (b_{xx}(t))^T p_0(t) + (\sigma_{xx}(t))^T q_0(t)]dt - Q_0(t)dW(t), \\ P_0(T) = \mu\varphi_{xx}(\bar{x}(T), \bar{y}(0)), \end{array} \right. \quad (4.11)$$

$$\left\{ \begin{array}{l} d\gamma(t) = f_y(t)\gamma(t)dt + f_z(t)\gamma(t)dW(t), \\ \gamma(0) = \lambda + \mu E[\varphi_y(\bar{x}(T), \bar{y}(0))]. \end{array} \right. \quad (4.12)$$

Thus we get the following theorem.

**Theorem 4.1** *Suppose (A1), (A2) and (A3) hold. Let  $\bar{u}(\cdot)$  be an optimal control with state constraint (4.1) and  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$  be the corresponding solution. Then there exist two constants  $\lambda, \mu$  with  $|\lambda|^2 + |\mu|^2 = 1$  such that*

$$\begin{aligned} & \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), u, \bar{x}(t), \bar{u}(t), p_0(t), q_0(t), P_0(t), p(t), q(t), P(t), \gamma(t)) \\ & \geq \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}, \bar{x}(t), \bar{u}(t), p_0(t), q_0(t), P_0(t), p(t), q(t), P(t), \gamma(t)), \end{aligned}$$

$$\forall u \in U, \text{ a.e., a.s.,}$$

where  $\mathcal{H}(\cdot), (p(\cdot), q(\cdot)), (P(\cdot), Q(\cdot)), (p_0(\cdot), q_0(\cdot)), (P_0(\cdot), Q_0(\cdot))$  and  $\gamma(\cdot)$  are defined in (4.9), (3.1), (3.2), (4.10), (4.11) and (4.12).

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